The Pill Problem and Catalan Numbers (MathFest 2009) 7/31/09
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The Pill Problem: Suppose you have a bottle of pills, and you want a daily dose of half a pill. Each day, remove a pill from the bottle. If it is a half pill, swallow it. If it is a whole pill, break it in half, swallow one half, and return the other half to the bottle. Over time, you will have a mix of whole and half pills in the bottle.

Question: After n days, what is the probability that the pill you remove from bottle is whole?

Kaleb Waite and I studied the above question in 2008. To help visualize all possible ways to exhaust the pills in the bottle, we introduced what we called the pill tree. The pill tree for 2 whole and 1 half pills is shown below:

(2,1)

(1,2) (2,0)

(0,3) (1,1) (1,1)

(0,2) (0,2) (1,0) (0,2) (1,0)

(0,1) (0,1) (0,1) (0,1) (0,1)

(0,0) (0,0) (0,0) (0,0) (0,0)

The pill problem is computationally difficult, because the number of nodes in the pill tree is very large, even for reasonably small initial numbers of whole and half pills.

New Question: Is there a formula for the number of nodes in the pill tree?

Let w and h denote the initial number of whole and half pills in the bottle, and let $T(w,h)$ denote the number of nodes in pill tree. Then T satisfies the following recurrence relation:

$$T(0,h) = h+1$$

$$T(w,0) = 1 + T(w-1,1)$$

$$T(w,h) = 1 + T(w-1,h+1) + T(w,h-1)$$

Margaret Bayer (bayer@math.ku.edu) and I were able to prove the following:

Theorem: $T(w,0) = \sum_{k=1}^{w+1} C_k$, where C_k denotes the k-th Catalan number.

Recall that C_k can be defined as the number of lattice paths from $(0,0)$ to (k,k) that do not go above the diagonal.

This result can be rephrased without mentioning pills. The pill tree starting with w whole pills and no half pills is isomorphic to the tree consisting of all lattice paths from $(0,0)$ to (w,w). Thus, the tree consisting of lattice paths described in the definition of Catalan numbers has an interesting property: The number of nodes in this tree is a sum of Catalan numbers.

Here are two immediate consequences of the theorem:

Corollary: $T(w,1)=\sum_{k=2}^{w+2} C_k$ and $T(w,2)=C_{w+3}-2$.

Finally, the theorem can be extended to a general formula for T:

Theorem: $T(w,h)=\sum_{k=h+1}^{w+h+1} \frac{h+1}{k+1}\binom{2k-h}{k-h}$.

For $h=2$, we have $C_{w+3}-2=T(w,2)=\sum_{k=3}^{w+3} \frac{3}{k+1}\binom{2k-2}{k-2}$.

After a substitution and a little algebra, the above boils down to the following: For $n \geq 2$,

$$C_n = 1+\sum_{k=2}^{n} \frac{3}{k+1}\binom{2k-2}{k-2} = 1+\frac{3}{3}\binom{2}{0}+\frac{3}{4}\binom{4}{1}+\frac{3}{5}\binom{6}{2}+\cdots+\frac{3}{n+1}\binom{2n-2}{n-2}.$$

Written recursively, this is

$$C_1 = 1$$

$$C_n = C_{n-1}+\frac{3}{n+1}\binom{2n-2}{n-2} = C_{n-1}+\frac{3}{n+1}(n-1)C_{n-1} = \left(1+\frac{3(n-1)}{n+1}\right)C_{n-1} = \frac{2(2n-1)}{n+1}C_{n-1}.$$

Note: In 1991, Knuth and McCarthy posed a different pill problem with the same setup. They asked for the expected number of half pills remaining when the whole pills run out.

References:

(Brandt and Waite) Using Recursion to Solve the Pill Problem, Journal of Computing Sciences in Colleges, 2009.

(Knuth and McCarthy) Big Pills and Little Pills, Problem E 3429, Amer. Math. Monthly, 1992.

(Koshy) Catalan Numbers with Applications, Oxford, 2009.