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Bohr's Radius for Polynomials in One Complex Variable

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Abstract. We consider complex polynomials of degree n that are bounded by one in the unit disc and give estimates on the size of the radius R_n of the disc where the sum of the moduli of the individual terms of the polynomial is less than one. We find that there are positive constants C_1, C_2 such that

$$C_1 \frac{1}{3^{n/2}} < R_n - \frac{1}{3} < C_2 \frac{\log n}{n}.$$

This result generalizes the celebrated theorem of Harald Bohr to polynomials of degree n.

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1. Introduction

We will write $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\|f\|_\infty := \sup_{|z|<1} |f(z)|$ throughout this paper. In a paper [8] compiled by Hardy in 1914 from his correspondence with Bohr, the following theorem is proved.

Theorem 1 (Bohr, 1914). *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* $\mathbb{D}$ *and* $\|f\|_\infty \le 1$*, then* $\sum_{k=0}^{\infty} |a_k| r^k \le 1$ *for* $r \le 1/3$*, and the constant* $1/3$*, often called the Bohr radius, cannot be improved.*

Actually, Bohr proved a weaker version of the result, that if $f(z)$ is analytic in $\mathbb{D}$ and $\|f\|_\infty \le 1$, then $\sum_{k=0}^{\infty} |a_k| r^k \le 1$ in a disc of radius $R \le 1/6$. That Bohr's radius is equal to $1/3$ is due to F. Wiener, M. Riesz and I. Schur, who solved the problem independently. Bohr's and Wiener's proofs can be found in [8].

In 1997, Boas and Khavinson generalized Bohr's Theorem to several complex variables [7]. They found bounds for Bohr's radius in any complete Reinhard domain and showed that the radius decreases to zero as the dimension of the domain increases. Their paper stimulated the current wave of research in Bohr

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type questions. Among others, Aizenberg, Aytuna, Djakov & Tarkhanov, and Defant & Frerick have obtained further results on Bohr phenomena in $\mathbb{C}^n$ (see [1, 2, 3, 9]). Recently, Beneteau, Dahlner, Khavinson and Korenblum (in [4, 5]) have investigated the existence of a Bohr radius in Hardy spaces. For a survey of various other questions and related results the reader may refer to Boas [6].

Interestingly, Dixon [10] drew attention to applications of Bohr phenomena to operator theory showing that Bohr's Theorem is useful in the characterization of Banach algebras that satisfy von Neuman's inequality. Paulsen, Popescu & Singh in [11] continue this line of research.

It is clear that R_n converges to $1/3$ as $n \to \infty$. This follows from Bohr's result. In this paper we study the asymptotics of Bohr radii for the class of polynomials

$$\Pi_n := \left\{ P_n(z) = \sum_{k=0}^{n} a_k z^k : \, a_n \neq 0, \|P_n\|_\infty = 1 \right\}.$$

Definition. We shall call R_n the *Bohr radius* for Π_n if $\sum_{k=0}^{n} |a_k| r^k < 1$ for $r < R_n$ and all $P_n \in \Pi_n$, and there exists P_n such that $\sum_{k=0}^{n} |a_k| {R_k}^k = 1$, i.e. the inequality is sharp.

In the next section we state our main result and outline its proof, leaving the proofs of the intermediate lemmas to a later section.

2. The main theorem

Theorem 2. *If $P_n \in \Pi_n$ and R_n is Bohr's radius for Π_n, there exist constants C_1, C_2 with $0 < C_1, C_2 < \infty$ such that*

$$C_1 \frac{1}{3^{n/2}} < R_n - \frac{1}{3} < C_2 \frac{\log n}{n} \tag{1}$$

for all $n \in \mathbb{N}$.

To prove the first inequality in (1) we will need two known auxiliary facts.

Lemma 1 (Wiener's Lemma [8])**.** *If the power series $\sum_{k=0}^{\infty} a_k z^k$ converges in $\mathbb{D}$ and $|\sum_{k=0}^{\infty} a_k z^k| \leq 1$ for $|z| < 1$, then $|a_k| \leq 1 - |a_0|^2$ for all $k \in \mathbb{N}$.*

Lemma 2 (Van der Corput and Visser's Inequality [12, Theorem A])**.** *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n such that $|p(z)| \leq M$ for $|z| \leq 1$. If a_u, a_v, $u < v$, are two coefficients such that for no other coefficient $a_w \neq 0$, we have $w \equiv u \mod (v-u)$, then $|a_u| + |a_v| \leq M$.*

Proof of Theorem 2. Fix $n \in \mathbb{N}$ and consider $P_n \in \Pi_n$. Without loss of generality we may assume that $0 < a_0 < 1$. We shall first prove the left inequality in (1).

Putting $M = 1$ and $u = 0$ in Lemma 2 we obtain the following statement.

If $\sum_{k=0}^{n} a_k z^k$ *is such that* $\left|\sum_{k=0}^{n} a_k z^k\right| < 1$ *for* $|z| < 1$*, then* $|a_k| \le 1 - a_0$ *for* $k > n/2$.

Assuming for simplicity that n is even and writing $|z| =: r$ the above statement together with Lemma 1 gives

$$\begin{aligned}\sum_{k=0}^{n} |a_k| r^k &\le a_0 + (1-{a_0}^2)\sum_{k=1}^{n/2} r^k + (1-a_0)\sum_{k=n/2+1}^{n} r^k \\ &< a_0 + 2(1-a_0)\frac{r - r^{n/2+1}}{1-r} + (1-a_0)\frac{r^{n/2+1} - r^{n+1}}{1-r} \\ &= a_0 + (1-a_0)\frac{2r - r^{n/2+1} - r^{n+1}}{1-r}.\end{aligned}$$

So $\sum_{k=0}^{n} |a_k| r^k < 1$ when $3r - r^{n/2+1} - r^{n+1} \le 1$.

Let us assume that the following lemma (which will be proved in Section 3) holds.

Lemma 3. *The equation* $3r - r^{n/2+1} - r^{n+1} = 1$ *has a unique solution* r_n *in the interval* $(1/3, 1)$.

Then it follows from the above lemma that Bohr's radius R_n is greater than or equal to r_n for all $n \in \mathbb{N}$. Moreover,

$$r_n - \frac{1}{3} = \frac{{r_n}^{n/2+1} + {r_n}^{n+1}}{3}$$

and, since $r_n > 1/3$ we have

$$R_n - \frac{1}{3} \ge r_n - \frac{1}{3} \ge \frac{C_1}{3^{n/2}}.$$

This completes the proof of the first inequality in (1). We are left to establish the second one, namely

$$R_n < \frac{1}{3} + C_2 \frac{\log n}{n}.$$

Fix a with $0 < a < 1$ and consider Wiener's function

$$f_a(z) = \frac{1-z}{1-az} = 1 + (a-1)z + (a^2 - a)z^2 + \cdots = \sum_{k=0}^{\infty} a_k z^k.$$

Note that, since $f_a(z)$ maps the closed unit disc $\overline{\mathbb{D}}$ onto

$$\left| z - \frac{1}{1+a} \right| \le \frac{1}{1+a},$$

we have

$$\|f_a(z)\|_\infty = f_a(-1) = \frac{2}{1+a}.$$

Wiener used this function to prove that Bohr's radius is no larger than 1/3. He showed that

$$\sum_{k=0}^{\infty}|a_k|r^k = 1+\frac{(1-a)r}{1-ar} > \frac{2}{1+a}$$

for $r > 1/(1+2a)$. We follow his approach in spirit and define the Wiener polynomial by

$$\begin{aligned} P_{n,a}(z) := \sum_{k=0}^{n} a_k z^k &= 1+(a-1)z+(a^2-a)z^2+\cdots+(a^n-a^{n-1})z^n \\ &= \frac{1-z}{1-az}-a^n z^{n+1}\frac{(a-1)}{1-az} = f_a(z)+a^n z^{n+1}\frac{(1-a)}{1-az}. \end{aligned}$$

The sequence of polynomials $P_{n,a}$ converges to f_a uniformly on $\mathbb{D}$ as $n \to \infty$ since

$$a^n z^{n+1}\frac{(1-a)}{1-az} \to 0$$

as $n \to \infty$ for a fixed a with $0 < a < 1$.

Now we apply the following lemma which will also be proved in Section 3.

Lemma 4. *For fixed a with $0 < a < 1$, if $a^n < \pi^2/(n+1)^2$ there exists $\hat{n}(a) \in \mathbb{N}$ such that*

$$\|P_{n,a}\|_\infty \le \frac{2}{1+a}$$

for all even $n \ge \hat{n}$.

Assuming this, let $M_{n,a}$ be the majorant function for $P_{n,a}$ defined by

$$\begin{aligned} M_{n,a}(r) := \sum_{k=0}^{n}|a_k|r^k &= 1+(1-a)r+\cdots+a^{n-1}(1-a)r^n \\ &= 1+\frac{(1-a)r}{1-ar}-\frac{1-a}{1-ar}a^n r^{n+1}. \end{aligned}$$

Then, for even $n \ge \hat{n}$,

$$M_{n,a} > \|P_{n,a}\|_\infty,$$

or

$$1+\frac{(1-a)r}{1-ar}-\frac{1-a}{1-ar}a^n r^{n+1} > \frac{2}{1+a},$$

for

$$r > \frac{1}{1+2a}+\frac{1+a}{1+2a}a^n r^{n+1}.$$

The following lemma (which also will be proved later in Section 3) allows us to handle the right hand side of this estimate.

Lemma 5. *The equation*

$$x = \frac{1}{1+2a} + \frac{1+a}{1+2a} a^n x^{n+1}$$

has a unique solution R_a *in the interval* $(1/3, 1)$ *for any fixed* a *with* $0 < a < 1$ *and any* $n \in \mathbb{N}$.

By this lemma and by Lemma 4 we have, that for a fixed and any even $n \geq \hat{n}$, $r > R_a$ implies $M_{n,a} > \|P_{n,a}\|_\infty$. Hence Bohr's radius R_n is less than R_a for all a. That is,

$$R_n < \frac{1}{1+2a} + \frac{1+a}{1+2a} a^n {R_a}^{n+1} = \frac{1}{3} + \frac{2}{3}\frac{1-a}{1+2a} + \frac{1+a}{1+2a} a^n {R_a}^{n+1}.$$

Moreover, from Lemma 4 we see that

$$a^n < \left(\frac{\pi}{n+1}\right)^2$$

implies

$$\|P_{n,a}\|_\infty \leq \frac{2}{1+a}$$

if

$$1 - a > 2\frac{\log n}{n}.$$

Hence, writing $1 - a = (c \log n)/n$ with some constant $c > 2$ yields

$$\frac{2}{3}\frac{1-a}{1+2a} < c_1 \frac{\log n}{n}$$

for some positive constant c_1. Also,

$$\frac{1+a}{1+2a} a^n {R_a}^{n+1} \frac{n}{\log n} \leq \left(\frac{\pi}{n+1}\right)^2 {R_a}^{n+1} \frac{n}{\log n} \to 0$$

as $n \to \infty$. Therefore there exists c_2 such that

$$\frac{1+a}{1+2a} a^n {R_a}^{n+1} < c_2 \frac{\log n}{n}.$$

Putting $C_2 = c_1 + c_2$ we get that

$$R_n < \frac{1}{3} + C_2 \frac{\log n}{n}$$

for all even $n \geq \hat{n}$. In fact, this holds for all $n \geq \hat{n}$ since R_n decreases to $1/3$ as $n \to \infty$.

This completes the proof of Theorem 1. ■

3. Proofs of lemmas

Proof of Lemma 3. Fix $n > 1$ and let

$$g(r) := 3r - r^{n/2+1} - r^{n+1} - 1.$$

Obviously, one zero for this equation is $r = 1$. We will show that g has a unique zero in the interval $(1/3, 1)$.

Notice that

$$g(\tfrac{1}{3}) = -\left[\left(\frac{1}{3}\right)^{n/2+1} + \left(\frac{1}{3}\right)^{n+1}\right] < 0, \qquad \text{and} \qquad g(1) = 0.$$

Furthermore, from

$$g'(r) = 3 - \left(\frac{n}{2} + 1\right) r^{n/2} - (n+1)r^n$$

we conclude $g'(1/3) > 0$ and $g'(1) = 1 - (3n)/2 < 0$ for $n > 1$.

Moreover

$$g''(r) = -\frac{n}{2}\left(\frac{n}{2} + 1\right) r^{n/2-1} - n(n+1)r^{n-1}$$

which implies that $g''(r) < 0$ for all $n > 1$. Hence, there is a $b \in (1/3, 1)$ where $g'(b) = 0$ and such that $g(b) > 0$. Thus, there is one zero for g in $(1/3, b)$, since the function is strictly increasing on that interval. And there are no zeros on the interval $(b, 1)$; the function is decreasing there and $g(1) = 0$. ■

Proof of Lemma 4. Recall that Wiener's polynomial is defined by

$$P_{n,a}(z) = \frac{1-z}{1-az} - a^n z^{n+1}\frac{(a-1)}{1-az} = f_a(z) + a^n z^{n+1}\frac{(1-a)}{1-az}.$$

Let $g(z) := a^n z^{n+1}$, then g is $2\pi/(n+1)$-periodic. First notice that for even $n \in \mathbb{N}$,

$$|P_{n,a}(-1)| = |f_a(-1) - a^n\frac{(1-a)}{1+a}| = \frac{2 - a^n(1-a)}{1+a} \nearrow \frac{2}{1+a} \qquad \text{as } n \to \infty$$

where $2/(1+a) = \|f_a(z)\|_\infty = f(-1)$. Now, by the periodicity of g, for any $\tau \in [0, 2\pi)$ there exists $\theta_0 \in [(n-1)/(n+1)\pi, \pi]$ such that

$$|P_{n,a}(e^{i\tau})| \leq |f_a(e^{i\tau})| + |g(e^{i\theta_0})| \leq |f_a(e^{i\theta_0})| + |g(e^{i\theta_0})|.$$

Hence, if

$$|f_a(e^{i\theta})| + |g(e^{i\theta})| \leq \frac{2}{1+a}$$

for all $\theta \in [(n-1)/(n+1)\pi, \pi]$ then

$$\|P_{n,a}\|_\infty \leq \frac{2}{1+a}.$$

Let $\beta_n := (n-1)/(n+1)$, $n > 2$, and let $\phi := \arg[f_a(e^{\beta_n \pi i})]$, then

$$P_{n,a}(e^{\beta_n \pi i}) = f_a(e^{\beta_n \pi i}) - a^n \frac{(1-a)}{1-ae^{\beta_n \pi i}},$$

and

$$\left|f_a(e^{\beta_n \pi i})\right| = \frac{2}{1+a}\cos\phi = \frac{2}{1+a}\frac{1-\cos\beta_n\pi}{\sqrt{2-2\cos\beta_n\pi}} = \frac{\sqrt{2(1-\cos\beta_n\pi)}}{1+a}.$$

Thus, if we can find $\hat{n}(a) \in \mathbb{N}$, such that

$$a^{\hat{n}} < \frac{2}{1+a} - |f_a(e^{\beta_{\hat{n}}\pi i})|,$$

then

$$\left|P_{n,a}(e^{\beta_n \pi i})\right| = \left|f_a(e^{\beta_n \pi i}) - a^n\frac{(1-a)}{1+a}\right| \le \left|f_a(e^{\beta_n \pi i})\right| + \left|a^n\frac{(1-a)}{1+a}\right| < \frac{2}{1+a}$$

for all even $n \ge \hat{n}$.

We shall now prove the following claim: *there exists* $\hat{n}(a) \in \mathbb{N}$ *such that, whenever* $n \ge \hat{n}$

$$a^n < \frac{2}{1+a} - \frac{\sqrt{2(1-\cos\beta_n\pi)}}{1+a}$$

or, equivalently,

$$1+\cos\beta_n\pi > a^n(1+a)\frac{4-(1+a)a^n}{2}.$$

In fact, since

$$\begin{aligned} -\cos\beta_n\pi &= \cos\left(\pi - \frac{n-1}{n+1}\pi\right) = \cos\left(\frac{2\pi}{n+1}\right) \\ &= 1 - \frac{4\pi^2}{(n+1)^2} + \mathcal{O}\left(\frac{1}{(n+1)^4}\right) \end{aligned}$$

and

$$4a^n > a^n(1+a)\frac{4-(1+a)a^n}{2},$$

we have that if $1+\cos\beta_n\pi > 4a^n$ then

$$1+\cos\beta_n\pi > a^n(1+a)\frac{4-(1+a)a^n}{2}.$$

Yet,

$$a^n < \frac{\pi^2}{(n+1)^2} + \mathcal{O}\left(\frac{1}{(n+1)^4}\right)$$

is obviously the case for n large enough. Let $\hat{n}$ be the smallest such n. This completes the proof of the claim.

Now,

$$\left|P_{n,a}(e^{\alpha\beta_n \pi i})\right| < \frac{2}{1+a}$$

for $0 < \alpha \le 1$ and for all even $n \ge \hat{n}$, since

$$1 - 4a^n > -\cos\beta_n\pi \ge -\cos\alpha\beta_n\pi.$$

Hence

$$\|P_{n,a}\|_\infty \le \frac{2}{1+a}.$$

■

Proof of Lemma 5. Define

$$h(x) := \frac{1}{1+2a} + \frac{1+a}{1+2a}a^n x^{n+1} - x.$$

For all $a \in (0,1)$ and all $n \in \mathbb{N}$ we have

$$\begin{aligned} h(\tfrac{1}{3}) &= \frac{1}{1+2a} + \frac{1+a}{1+2a}a^n\left(\frac{1}{3}\right)^{n+1} - \frac{1}{3} > \frac{1+a}{1+2a}a^n\left(\frac{1}{3}\right)^{n+1} > 0, \\ h(1) &= \frac{(1+a)a^n - 2a}{(1+2a)} < 0 \end{aligned}$$

because $n > \log_a(2a/(1+a))$ for all n.

Now, the function h is strictly convex since

$$h''(x) = \frac{(1+a)a^n n(n+1)}{1+2a}x^{n-1} > 0$$

for all x. Therefore, by the Intermediate Value Theorem, there is a zero for h in $(1/3, 1)$ and that zero is unique because the function is decreasing on that interval. ■

4. Remarks

The original conjecture we set off to prove was that Bohr's radius for polynomials would converge to 1/3 at a geometric rate. We are unable to prove this. In fact, there is no maximal function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $\|f\|_\infty = 1$ and such that $\sum_{k=0}^{\infty} |a_n|(1/3)^k = 1$. As Khavinson has pointed out, suppose such a function exists, then

$$1 = \sum_{k=0}^{\infty} |a_k|\frac{1}{3^k} \le a_0 + (1 - {a_0}^2)\sum_{k=1}^{\infty}\frac{1}{3^k}$$

or, after some algebra, $a_0 \ge 1$, which is a contradiction. Thus, Wiener's polynomials are the only candidates for extremals in the right hand side of (1). Yet, as our argument shows, they are insufficient to yield geometric convergence. There is definitely plenty of room for improvement in the bounds of the estimates. It would also be interesting to explore how Bohr radii behave for polynomials in several complex variables, in different spaces and with different norms.

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